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POLARIZATION RADIATION IN THE PLANETARY ATMOSPHERE DELIMITED BY A HETEROGENEOUS DIFFUSELY REFLECTING SURFACE

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16. Abstract

Spatial-frequency characteristics (SFC) and the scattering function of a layer which is uniform on the horizontal with absolutely black bottom, or an isolated layer, are the basis for a mathematical model which describes the horizontal heterogeneities in a light field with regard for radiation polarization in a three-dimensional planar atmosphere, delimited by a heterogeneous surface with diffuse reflection. The perturbation method was used to obtain vector transfer equations which correspond to the linear and nonlinear systems of polarization radiation transfer. The boundary value tasks for the vector transfer equation that is a parametric set and one dimensional are satisfied by the SFC of the nonlinear system, and are expressed through the SFC of linear approximation. As a consequence of the developed theory, formulas were obtained for analytical calculation of albedo in solving the task of dissemination of polarization radiation in the planetary atmosphere with uniform Lambert bottom.

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Key words: vector transfer equation, polarization radiation, three-dimensional planar layer, heterogeneous albedo, spatial-frequency characteristics, scattering functions, fundamental solution, linear and nonlinear system.

POLARIZATION RADIATION IN THE PLANETARY ATMOSPHERE DELIMITED BY A HETEROGENEOUS DIFFUSELY REFLECTING SURFACE

By S. A. Strelkov and T. A. Sushkevich

Introduction /3*

Study of the polarization characteristics of planetary atmospheric radiation has been underway already for over 100 years, if we even begin to count from the founding work [1], whose author for the first time proposed the four parameters which bear his name to describe the partially polarized Stokes light-vector. Calculation of polarization radiation in repeatedly scattering heterogeneous, but even isotropic media with non-Rayleigh law of radiation interaction with substance, until now has presented great computational difficulties, although a lot of attention of many specialists of different countries using the latest achievements in the field of modern numerical and analytical methods, and also the most powerful computers has been focused on these problems in the last 30 years. At the same time, a trend has been observed towards expanding the sphere of application of the polarization radiation in studies by specialists working in different fields, including astrophysics, atmospheric and space optics, hydrooptics, radio physics, ecology (environmental pollution), remote sensing of the earth's surface, ocean, seas, reservoirs, etc., in relation to studying natural resources. Naturally it is necessary to develop methods which allow calculation of the effect of heterogeneous underlying surfaces on the formation of the polarized radiation field of the medium. Since solution to the transfer equation with regard for polarization in itself is a cumbersome task, it is necessary to develop a /4 method for calculating the diffuse reflection from the heterogeneous underlying surface that does not result in additional complications and considerable increase in the volume of the computations.

This publication suggests a mathematical model which corresponds to these requirements. It is based on one-dimensional vector kinetic equations for spatial-frequency characteristics (SFC) of a system of polarized

*Numbers in margin indicate pagination in original foreign text.

radiation transfer, invariant in relation to the constant component and horizontal variations in the underlying surface albedo and the conditions of layer illumination. The method of integrated Fourier transform was used to construct the fundamental solutions, or the functions of the influence [2] of a three-dimensional vector transfer equation with perturbed boundary condition. In the theory of vision in turbid media, and in the tasks of Fourier optics, the functions of influence are usually called scattering functions [3, 4]. The Stokes vector is presented in the form of an integrated packet of fundamental solutions or SFC with horizontal variations in the albedo of the underlying surface by the method of summing the perturbations. These results are the development and generalization for the vector task of the results presented in publication [5, 6].

The formula for calculating the albedo of a uniform Lambert surface, depolarizing the radiation in the atmosphere follows from a constructed solution as a particular case. This correlation permits a significant reduction in the volume of computations, since calculation of the Stokes vector is required to solve the task of calculating the Stokes vector in the atmosphere-reflecting underlying surface system only for an isolated heterogeneous atmosphere. This formula was derived for the first time and is a generalization for the vector case of the formula for a scalar task of the transfer theory previously obtained by V. V. Sobolev [7, 8], Van de Holst [12], Z. G. Yanovitskiy [9], I. N. Minin [10], V. V. Ivanov [11] and the authors of publication [13].

In order to study the link between polarization radiation reflected by the planet, the characteristics of diffuse reflection of its surface, an expression was found for direct optical transfer operator of the atmosphere, generalization of the vector task for the result presented in publication [13].

1. Mathematical Statement of the Task

An external parallel stream of radiation $\pi \vec{S}_{\lambda}$ falls on the upper boundary of a planar layer unlimited in horizontal coordinates and finite

on the vertical ($0 \le z \le H$, H-geometric thickness of the layer) at angle $\bar{\vartheta}_0 = \arccos \mu_0$ to the perpendicular to the layer with azimuth φ_0 , $\vec{s}_0 = \{\mu_0, \varphi_0\}$. The vector $\vec{t}$ describes the condition of polarization of the incident light streams; $\vec{t} = \{1,0,0,0\}$ corresponds to nonpolarized, natural light. πS_λ is the power of the stream, λ is the index of the wavelength of monochromatic radiation. Direction of dissemination of radiation $\vec{s}$ is described by spherical coordinates: $\vartheta = \arccos \mu$, $\vartheta \in [0,\pi]$ -- zenith angle which is counted from the direction of the inner perpendicular to the upper layer boundary $z = 0$. It coincides with the axis z, and $\varphi \in [0,2\pi]$ --azimuth counted from the positive direction of the x-axis. We adopt as $\varphi = 0$, the azimuth of the external monodirectional stream $\varphi_0 = 0$. The values $\mu = \mu^+ \in (0, 1)$ correspond to the descending (transmitted) radiation, while $\mu = \mu^- \in [-1,0)$ --ascending (reflected) radiation. The single sphere $\Omega = [-1,1] \times [0,2\pi]$ --set of all the directions $\vec{s} = \{\mu,\varphi\}$ with $\mu \in [-1,1]$ and $\varphi \in [0,2\pi]$; $\Omega^+ = [0,1] \times [0,2\pi]$ --hemisphere for the descending radiation; $\Omega^- = [-1,0] \times [0,2\pi]$ --hemisphere for the ascending radiation.

In order to simplify writing everywhere, below we will omit the vector magnitude sign ("$\rightarrow$" on the top) of the direction vectors (for example $s = \{s_\perp,\mu\} = \vec{s} = \{\vec{s}_\perp,\mu\}$, $s' = \vec{s}'$; $s_0 = \vec{s}_0$, $s_\perp = \vec{s}_\perp = \{\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi\}$ and the coordinates (for example $r = \vec{r} = \{x,y,z\} = \{\vec{r}_\perp, z\}$, $r_\perp = \vec{r}_\perp = \{x,y\}$). Below we will also not constantly mention that $-\infty < x, y < \infty$. The integrals on the single sphere or the hemispheres will be written as:

$$\int_\Omega f(s')ds' = \int_0^{2\pi} d\varphi' \int_{-1}^{1} f(\mu',\varphi')d\mu'; \quad \int_{\Omega^+} f(s')ds' = \int_0^{2\pi} d\varphi' \int_0^1 f(\mu',\varphi')d\mu'.$$

In order to reduce the writing in the boundary conditions of the boundary value task, we will write $\Phi|_0$ instead of $\Phi(r_\perp,z,s)|_{z=0}$ and $\Phi|_H$ instead of $\Phi(r_\perp,z,s)|_{z=H}$. With integrated differential equations of transfer in the boundary value task, we will omit description of the area of their analysis, recalling that everywhere it will be the same:

$[z \in (0,H), s \in \Omega; z = 0, s \in \Omega^+; z = H, s \in \Omega^-]$.

On the hypothesis of stationary state of the medium and constant external radiation flow, the field of the quasimonochromatic polarized

radiation is completely described by a four dimensional vector $\vec{\Phi}_{tot}(r, s)$ Its components in the SP-presentation [14] are Stokes parameters $\vec{\Phi}_{tot}(r,s)=\{I_{tot}(r,s), Q_{tot}(r,s), U_{tot}(r,s), V_{tot}(r,s)\}$, which have dimensionality of radiation intensity [15, 16, 17]. These parameters are used to reflect all the characteristics of radiation polarization, and the first component $I_{tot}(r,s)$ coincides with the energy brightness, or complete intensity of radiation at the point in space with coordinate r in the direction s.

If the medium is macroscopically optically isotropic, i.e., there are no phenomena of linear and circular double beam refraction and dichroism, then the matrix describing the photon scattering law on the particle with regard for polarization of the light wave $\hat{P}(z, s' \to s)$ is a function of the scattering angle x [18 - 21]: /6

$$\hat{P}(z,s'\to s)=\hat{P}(z,\hat{ss'})=\hat{P}(z,\cos\chi)=\hat{P}(z,\mu,\mu',\varphi'-\varphi)=[P_{ij}(z,\cos\chi)],\quad i,j=1,2,3,4,$$

where

$$\cos\chi=\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\varphi-\varphi') .$$

We will examine the atmosphere adjoining the horizontally heterogeneous Lambert surface (task for planetary atmosphere with Lambert bottom). Reflection from the bottom is diffuse, while the reflected light is nonpolarized. The matrix of the Lambert reflection is $\hat{P}_H=[\delta(i-1)\delta(j-1)]$, $i,j=1,2,3,4,$ while the Lambert law for polarized radiation can be written as

$$\vec{\Phi}_{tot}/_H=[q_H(r_\perp)\hat{R}I_{tot}(r_\perp,H,s)]\vec{f}_0 \qquad (1)$$

where $\vec{f}_0=\{1,0,0,0\}$, i.e., $\vec{\Phi}_{tot}(r_\perp,H,s)=\{I_{tot}(r_\perp,H,s), 0, 0, 0\}$; the diffusely reflected intensity $I_{tot}(r_\perp,H,s)=q_H(r_\perp)\hat{R}I_{tot}$ at each point $r_\perp$ of the plane at level $z=H$ is isotropic for all directions $s\in\Omega^-$. The operator of reflection

$$\hat{R}I = \frac{1}{\pi}\int_{\Omega^+} I(\tau_\perp, H, s')\mu' ds'. \qquad (2)$$

Under these conditions complete Stokes vector function $\vec{\Phi}_{tot}(\tau, s)$ is solution to the following vector boundary value task 1: (3) - (5).

$$\hat{D}\vec{\Phi}_{tot} = \hat{S}\vec{\Phi}_{tot};\quad \vec{\Phi}_{tot}\big|_0 = \pi S_\lambda \delta(s-s_0)\vec{i};\quad \vec{\Phi}_{tot}\big|_H = (q_H(\tau_\perp)\hat{R}I_{tot})\vec{i}_H.$$

$\delta(x)$ --here and everywhere below is the Dirac function.

The transfer operator

$$\hat{D}\vec{\Phi} = \mu\frac{\partial\vec{\Phi}}{\partial z} + \left(s_\perp, \frac{\partial\vec{\Phi}}{\partial \tau_\perp}\right) + \sigma_{tot}(z)\vec{\Phi}(\tau, s), \qquad (6)$$

$$\left(s_\perp, \frac{\partial\vec{\Phi}}{\partial \tau_\perp}\right) = \sin\vartheta\sin\varphi\frac{\partial\vec{\Phi}}{\partial y} + \sin\vartheta\cos\varphi\frac{\partial\vec{\Phi}}{\partial x}. \qquad (7)$$

The integral of collisions, the function of the source

$$\hat{S}\vec{\Phi} = \sigma_{sc}(z)\int_{\Omega} \hat{P}(z,\mu,\mu',\varphi'-\varphi)\vec{\Phi}(\tau_\perp, z, \mu', \varphi')ds' \qquad (8)$$

$\sigma_{tot}(z)$ and $\sigma_{sc}(z)$ --altitude distributions of volumetric coefficients of the extinction and scattering respectively. The nucleus of the integral operator (8) is the angular matrix $\hat{P}(z,\mu,\mu',\varphi'-\varphi) = \hat{L}(\alpha)\hat{P}(z,\cos\gamma)\hat{L}(-\alpha')$, whose structure and properties are described in fair detail in publications [21 - 25]. We note merely that the matrix $\hat{P}$ in contrast to the scattering matrix $\hat{P}$ is not a function of the scattering angle, but depends on three angular variables: μ, μ' and $\varphi-\varphi'$. /7

The albedo of heterogeneous Lambert surface is presented in the form

$$q_H(z_1) = q + \varepsilon q_r(z_1) \leq 1, \tag{9}$$

where the parameter $0 \leq \varepsilon \leq 1$, $0 \leq q \leq 1$ -certain constant component which does not depend on the coordinates x, y; $q_r(z_1)$ describes the horizontal heterogeneities, $|q_r(z_1)| \leq 1$.

Thus, the task is solved of dissemination of radiation with regard for its polarization in a scattering and absorbing layer which is heterogeneous in height with horizontally nonuniform diffusely reflecting bottom (without multiplication).

The boundary value task 1 is linear, and the complete Stokes vector can be presented in the form of superposition of two components

$$\vec{\Phi}_{tot}(z,s) = \vec{\Phi}^{o}(z,s) + \vec{\Phi}(z,s). \tag{10}$$

The vector $\vec{\Phi}^{o} = \{I^{o}, Q^{o}, U^{o}, V^{o}\}$ corresponds to the nonscattering radiation from a monodirectional external source, and satisfies boundary value task 2:

$$\hat{D}_z \vec{\Phi}^{o} = 0; \quad \vec{\Phi}^{o}|_0 = \pi S_\lambda \delta(s-s_0)\vec{t}; \quad \vec{\Phi}^{o}|_H = 0, \tag{11 - 13}$$

where the one-dimensional differential transfer operator

$$\hat{D}_z = \mu \frac{\partial}{\partial z} + \sigma_{tot}(z), \tag{14}$$

and is computed analytically [23]:

$$\vec{\Phi}^{o}(z,s) = \left\{\pi S_\lambda e^{-\tau(z)/\mu_0} \delta(s-s_0)\vec{t}, \; s \in \Omega^+; \; 0, \; s \in \Omega^-\right\}. \tag{15}$$

The optical thickness at level z

$$\tau(z) = \int_0^z \sigma_{ext}(z')dz' \qquad (16)$$

The complete optical thickness of the layer $\tau_H = \tau(H)$.

Component $\vec{\Phi}(z,s) = \{I, Q, U, V\}$ corresponding to the Stokes vector with regard for repeated scattering in the layer and diffuse reflection from the bottom is found as a solution to boundary value task 3:

$$\hat{D}\vec{\Phi} = \hat{S}\vec{\Phi} + f_0\vec{E}, \quad \vec{\Phi}|_0 = 0, \quad \vec{\Phi}|_H = q_H(\tau_H)\left(\hat{R}I + f_H\right)\vec{E}_H, \qquad (17 - 19)$$

where

$$f_H = f_H(\mu_0) = \hat{R}I^0 = S_\lambda \mu_0 e^{-\tau_H/\mu_0}, \quad f_0\vec{E} = \hat{S}\vec{\Phi}^0; \qquad (20 - 22)$$
$$f_0 = f_0(z, \mu, \mu_0, \varphi_0 - \varphi) = \pi S_\lambda \sigma_{sc}(z) e^{-\tau(z)/\mu_0} \beta(z, \mu, \mu_0, \varphi_0 - \varphi)$$

2. Mathematical Model of Vector Spatial Frequency Characteristics

Solution to boundary value task 3 is found in the class of generalized functions which assume the existence of an integrated Fourier transform [2] in the form of a series of the perturbation theory for the parameter ε:

$$\vec{\Phi}(z, s) = \sum_{k=0}^{\infty} \varepsilon^k \vec{\Phi}_k(z, s), \quad \Phi = \{I, Q, U, V\} \qquad (23)$$

We substitute (23) and (9) into the boundary value task 3, and present for it the equivalent presentation, boundary value task 4:

$$\sum_{k=0}^{\infty} \varepsilon^k \hat{\mathcal{D}} \vec{\Phi}_k = \sum_{k=0}^{\infty} \varepsilon^k \hat{S} \vec{\Phi}_k + f_0 \vec{E},$$
$$\sum_{k=0}^{\infty} \varepsilon^k \vec{\Phi}_k \Big|_0 = 0,$$
$$\sum_{k=0}^{\infty} \varepsilon^k \vec{\Phi}_k \Big|_H = (q + \varepsilon q_r(\tau_\perp))(f_H + \sum_{k=0}^{\infty} \varepsilon^k \hat{R} I_k) \vec{t}_H, \qquad (24 - 26)$$

from which we will obtain boundary value tasks for different approximations of $\vec{\Phi}_k$ of the solution $\vec{\Phi}$ in the form of a series (23).

With k = 0, we obtain boundary value task 5 with coefficients and boundary values which do not depend on the coordinate $\tau_\perp$:

$$\hat{\mathcal{D}}_z \vec{\Phi}_0 = \hat{S} \vec{\Phi}_0 + f_0 \vec{E}; \quad \vec{\Phi}_0 \big|_0 = 0; \quad \vec{\Phi}_0 \big|_H = q E \vec{t}_H. \qquad (27 - 29)$$

This is a standard one-dimensional vector transfer equation whose solution yields a horizontally independent component of the Stokes vector $\vec{\Phi}_0 (z, s) = \{I_0, Q_0, U_0, V_0\}$, through which illumination πE is determined

$$E(s_0) = \hat{R} I_0 + f_H. \qquad (30)$$

The method, algorithms and programs for solving the boundary value task 4 and processing of the calculation results have been well studied, tested and presented in detail in publications [22 - 28]. Results of numerical solution to this task for a set of optical models of the atmosphere with underlying surface of calm ocean and cloud cover type are presented in publications [28 - 30] and reported at the all-union conference [31, 32]. The developed technique and the program system AP-5 [25] make it possible to calculate the Stokes vector with sufficiently high accuracy. With physically correct optical parameters of the atmospheric model, rapid convergence of the iterations and stability of the finite differential method are observed [28].

We see further development of the technique for solving the task of dissemination of radiation in planar media with regard for polarization

in the need for developing a method for calculating the diffused reflection from uniform and nonuniform underlying surface based on solution to a one-dimensional vector transfer equation for an insulated layer. This is possible with generalizing the SFC method [5, 6, 33, 34] for the vector task, i.e., with creation of a mathematical model of vector SFC.

We will introduce a designation for the Fourier transform operation /9 for the coordinate $r_\perp$:

$$\mathcal{F}[f](p) = \int_{-\infty}^{\infty} f(r_\perp) e^{i(p, r_\perp)} dr_\perp ; \quad dr_\perp = dx\,dy. \tag{31}$$

The f function can also have other arguments which differ from $r_\perp$. The spatial frequency $p = \{p_x, p_y\}$ only adopts real values. The inverse Fourier transform is defined as:

$$f(r_\perp) = \mathcal{F}^{-1}[f](r_\perp) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \mathcal{F}[f](p) e^{-i(p, r_\perp)} dp ; \quad dp = dp_x dp_y.$$

In order to reduce the writing, we will indicate everywhere one sign of the integral, recalling however, that integration is done by pairs of variables (x, y) or (P_x, P_y). We will introduce the spatial Fourier spectra: for albedo variations:

$$f(p) = \mathcal{F}[q_r](p) ; \tag{32}$$

for the Stokes vector component (we designate the Stokes vector components by printed letters, and the Fourier patterns by the corresponding hand written letters):

$$\mathcal{I}_\kappa(z, p_\kappa, s) = \mathcal{F}[I_\kappa](p_\kappa); \quad \mathcal{Q}(z, p_\kappa, s) = \mathcal{F}[Q_\kappa](p_\kappa);$$
$$\mathcal{U}_\kappa(z, p_\kappa, s) = \mathcal{F}[U_\kappa](p_\kappa); \quad \mathcal{V}(z, p_\kappa, s) = \mathcal{F}[V_\kappa](p_\kappa); \tag{33 - 36}$$

For the Stokes vector

$$\vec{\mathcal{P}}_\kappa(z, p_\kappa, s) = \mathcal{F}[\vec{\Phi}_\kappa](p_\kappa) = \{\mathcal{I}_\kappa, \mathcal{G}_\kappa, \mathcal{U}_\kappa, \mathcal{V}_\kappa\} \tag{37 - 38}$$

$\kappa \geqslant I$ --index of the expansion term (23).

The Fourier transform (31) applied to expression (6), results in a linear operator, which is one-dimensional in space:

$$\hat{K}(p)\vec{\mathcal{P}}(z,p,s)=\mu\frac{\partial\vec{\mathcal{P}}}{\partial z}+[\epsilon_{tot}(z)-i(p,s_\perp)]\vec{\mathcal{P}};\ \hat{K}(p=0)\equiv\hat{D}_z. \tag{39}$$

In this case we use the ratios [2]:

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial z}[\vec{\varphi}(r_\perp,z,s)e^{i(p,r_\perp)}]dr_\perp=\frac{\partial\vec{\mathcal{P}}}{\partial z};\ \int_{-\infty}^{\infty}\frac{\partial}{\partial r_\perp}[\vec{\varphi}e^{i(p,r_\perp)}]dr_\perp=-ip\vec{\mathcal{P}}(z,p,s)$$

The integrated operators of the function of source $\hat{S}$ and reflection $\hat{R}$ do not depend on the coordinates $r_\perp$, therefore

$$\hat{S}\vec{\mathcal{P}}=\mathcal{F}[\hat{S}\vec{\varphi}](p);\quad \hat{R}\mathcal{I}=\mathcal{F}[\hat{R}I](p).$$

With k = 1, from the boundary value task 4, we obtain boundary value task 6 for linear approximation of a number of perturbations (23):

$$\hat{D}\vec{\Phi}_1=\hat{S}\vec{\Phi}_1;\quad \vec{\Phi}_1|_0=0;\ \vec{\Phi}_1|_H=(q\hat{R}I_1+q_p(r_\perp)E)\vec{t}_H. \tag{42 - 44}$$

After Fourier transform (31), in boundary value task 4 we arrive at boundary value task 7:

$$\vec{\mathcal{P}}_1|_0=0,\quad \hat{K}(p_1)\vec{\mathcal{P}}_1(z,p_1,s)=\hat{S}\vec{\mathcal{P}}_1,\quad \vec{\mathcal{P}}_1|_H=(q\hat{R}\mathcal{I}_1+f(p_1)E)\vec{t}_H, \tag{45 - 47}$$

in which wave number P_1 is a parameter. We will attribute to wave numbers P, the index corresponding to the order of approximation in expansion (23). We note at once that all the boundary value tasks with differential operator $\hat{K}(p)$ are vector transfer equations with complex coefficient of extinction equal to $(\epsilon_{tot}-i(p,s_\perp))$ and consequently, the unknown functions will be complex. The numerical method of solving the transfer equation with complex function was developed [36]. Its generalization for the vector equation does not represent a problem, since it naturally is combined with the method developed for solving the standard vector transfer /10

equation [22 - 25, 28].

It is easy to show (for example, as done in publication [5]), that because of the linear nature of boundary value task 7, the Fourier patterns of vector $\vec{\Phi}_1$ and albedo q_r are linked by a linear relationship

$$\vec{\mathcal{P}}_1(z,p_1,s) = \vec{\Psi}_1(z,p_1,s)\,E\,f(p_1) \tag{48}$$

or

$$\begin{bmatrix} \mathcal{J}_1(z,p_1,s) \\ G_1(z,p_1,s) \\ \mathcal{U}_1(z,p_1,s) \\ \mathcal{V}_1(z,p_1,s) \end{bmatrix} = \begin{bmatrix} X_1(z,p_1,s) \\ Y_1(z,p_1,s) \\ Z_1(z,p_1,s) \\ T_1(z,p_1,s) \end{bmatrix} \cdot E\,f(p_1), \tag{49}$$

where the vector linear SFC

$$\vec{\Psi}_1(z,p_1,s) = \{X_1, Y_1, Z_1, T_1\} \tag{50}$$

satisfies boundary value task 8

$$\hat{K}(p_1)\vec{\Psi}_1 = \hat{S}\vec{\Psi}_1;\quad \vec{\Psi}_1\big|_0 = 0;\quad \vec{\Psi}_1\big|_H = (q\hat{R}X_1 + 1)\vec{E}_H \tag{51 - 53}$$

The vector $\vec{\Psi}_1$ does not depend on the conditions of illumination of the layer and variations in albedo $q_r(r_\perp)$. The inverse Fourier transform yields:

$$\vec{\Phi}_1(r_\perp,z,s) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\vec{\mathcal{P}}_1(z,p_1,s)\,e^{-i(p_1,r_\perp)}dp_1 = \frac{E}{(2\pi)^2}\int_{-\infty}^{\infty}\vec{\Psi}_1(z,p_1,s)\cdot f(p_1)\,e^{-i(p_1,r_\perp)}dp_1 = \frac{E}{(2\pi)^2}\int_{-\infty}^{\infty}\vec{\Psi}_1(z,p_1,s)\int_{-\infty}^{\infty} q_r(r_\perp')\,e^{i(p_1,r_\perp'-r_\perp)}dr_\perp'\,dp_1. \tag{54}$$

For the linear approximations of the series of perturbations (23) on the order $n \geq 2$, we obtain boundary value task 9 from boundary value task 4:

$$\hat{D}\vec{\Phi}_n = \hat{S}\vec{\Phi}_n;\quad \vec{\Phi}_n\big|_0 = 0;\quad \vec{\Phi}_n\big|_H = (q\hat{R}I_n + q_r(r_\perp)\hat{R}I_{n-1})\vec{E}_H, \tag{55 - 57}$$

in which two successive approximations are recurrently linked through the first component I_k of the vector $\vec{\Phi}_k$ which is included in the boundary condition. For nonlinear approximations, the following occurs

Theorem

If the vector function

$$\vec{\Psi}_k(z, p_k, \ldots, p_1, s) = \{X_k, Y_k, Z_k, T_k\}. \tag{58}$$

with fixed values of spatial frequencies $p_1, \ldots, p_k$ satisfies the set of parametric boundary value tasks 10 which are one dimensional in space: /11

$$\begin{aligned} &\hat{K}(p_k)\vec{\Psi}_k(z, p_k, \ldots, p_1, s) = \hat{S}\vec{\Psi}_k, \quad \vec{\Psi}_k\big|_0 = 0, \\ &\vec{\Psi}_k\big|_H = (q\hat{R}X_k(H, p_k, \ldots, p_1, S) + \hat{R}X_{k-1}(H, p_{k-1}, \ldots, p_1, s))\vec{t}_H, \end{aligned} \tag{59 - 61}$$

then the Fourier-patterns $\vec{\mathcal{P}}_k(z, p_k, s)$, the vector of the functions $\vec{\Phi}_k(z_k, z, s)$, being solutions to the corresponding boundary value task 9, are unequivocally defined through the vector function $\vec{\Psi}_k$ with the help of the integral ratios:

$$\vec{\mathcal{P}}_k(z, p_k, s) = \frac{E}{(2\pi)^{2(k-1)}} \int_{-\infty}^{\infty} \cdots \int f(p_1) f(p_2 - p_1) \cdots f(p_k - p_{k-1}) \vec{\Psi}_k \, dp_{k-1} \ldots dp_1. \tag{62}$$

Proof.

We will make the Fourier transform (31) in boundary value task 9, and will arrive at boundary value task 11:

$$\begin{aligned} &\hat{K}(p_k)\vec{\mathcal{P}}_k(z, p_k, s) = \hat{S}\vec{\mathcal{P}}_k; \\ &\vec{\mathcal{P}}_k\big|_0 = 0; \quad \vec{\mathcal{P}}_k\big|_H = (q\hat{R}\mathcal{I}_k(p_k) + \hat{R}\mathcal{F}[q_r I_{k-1}](p_k))\vec{t}_H. \end{aligned} \tag{63 - 65}$$

We will make an induction assumption, by introducing vector function $\vec{\Psi}_{k-1}$ so that the following ratio occurs

$$\vec{\mathcal{P}}_{k-1}(z, p_{k-1}, s) = \frac{E}{(2\pi)^{2(k-2)}} \int\cdots\int_{-\infty}^{\infty} f(p_1) f(p_2-p_1)\cdots f(p_{k-1}-p_{k-2}) \vec{\Psi}_{k-1}(z, p_{k-1},\dots,p_1,s)\, dp_{k-2}\cdots dp_1. \tag{66}$$

then

$$\begin{aligned} \mathcal{F}[q_p I_{k-1}](p_k) &= \int_{-\infty}^{\infty} q_p(\tau_1) I_{k-1}(\tau_1, z, s)\, e^{i(p_k,\tau_1)}\, d\tau_1 = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathcal{Y}_{k-1}(z,p_{k-1},s) \cdot \\ &\cdot \int_{-\infty}^{\infty} q_p(\tau_1)\, e^{i(p_k-p_{k-1},\tau_1)}\, d\tau_1\, dp_{k-1} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathcal{Y}_{k-1}(z,p_{k-1},s)\, f(p_k-p_{k-1}) \cdot \\ &\cdot dp_{k-1} = \frac{E}{(2\pi)^{2(k-1)}}\int\cdots\int_{-\infty}^{\infty} f(p_1) f(p_2-p_1)\cdots f(p_k-p_{k-1})\, X_{k-1}(z,p_{k-1},\dots,p_1,s) \cdot \\ &\cdot dp_{k-1}\cdots dp_1. \end{aligned} \tag{67}$$

We substitute (67) into (65):

$$\begin{aligned} &\hat{K}(p_k)\vec{\mathcal{P}}_k(z,p_k,s) = \hat{S}\vec{\mathcal{P}}_k; \quad \vec{\mathcal{P}}_k|_0 = 0; \quad \vec{\mathcal{P}}_k|_H = (q\hat{R}\mathcal{Y}_k(p_k) + \\ &+ \frac{E}{(2\pi)^{2(k-1)}}\int\cdots\int_{-\infty}^{\infty} f(p_1) f(p_2-p_1)\cdots f(p_k-p_{k-1})\,\hat{R}X_{k-1}(H,p_{k-1},\dots,p_1,s)\,dp_{k-1}\cdots \\ &\cdots dp_1)\,\vec{E}_H. \end{aligned} \tag{68 - 70}$$

Assume that the vector function $\vec{\Psi}_k(z, p_k, \dots, p_1, s)$ with fixed parameters $P_1,\dots,P_k$ satisfies the boundary value tasks 10. We multiply the /12 left and right sides of (59 - 61) by the product $\frac{E}{(2\pi)^{2(k-1)}} \cdot f(p_1) f(p_2-p_1)\cdots f(p_k-p_{k-1})$ and we integrate for parameters $P_{k-1},\dots,P_1$ in limits from $-\infty$ to $+\infty$. As a result, taking into consideration that operators $\hat{K}(p_k)$, $\hat{S}$ and $\hat{R}$ can be removed as signs of integrals for parameters, we obtain:

$$\left|\begin{aligned} &\hat{K}(p_k)\left[\frac{E}{(2\pi)^{2(k-1)}}\int\cdots\int_{-\infty}^{\infty} f(p_1) f(p_2-p_1)\cdots f(p_k-p_{k-1})\,\vec{\Psi}_k(z,p_k,\dots,p_1,s) \cdot \right. \\ &\left. dp_{k-1}\cdots dp_1\right] = \hat{S}\left[\frac{E}{(2\pi)^{2(k-1)}}\int\cdots\int_{-\infty}^{\infty} f(p_1) f(p_2-p_1)\cdots f(p_k-p_{k-1}) \cdot \right. \\ &\left. \cdot \vec{\Psi}_k(z,p_k,\dots,p_1,s)\, dp_{k-1}\cdots dp_1\right]; \\ &\frac{E}{(2\pi)^{2(k-1)}}\int\cdots\int_{-\infty}^{\infty} f(p_1) f(p_2-p_1)\cdots f(p_k-p_{k-1})\,\vec{\Psi}_k(z,p_k,\dots,p_1,s) \cdot \\ &\cdot dp_{k-1}\cdots dp_1\Big|_0 = 0; \end{aligned}\right. \tag{71 - 73}$$

$$\frac{E}{(2\pi)^{2(k-1)}}\int\limits_{-\infty}^{\infty}\!\!\cdots\!\!\int f(p_1)f(p_2-p_1)\cdots f(p_k-p_{k-1})\vec{\Psi}_k(z,p_k,\dots,p_1,s)\times$$
$$\times dp_{k-1}\cdots dp_1\Big|_H=\Big(q\hat{R}\Big[\frac{E}{(2\pi)^{2(k-1)}}\int\limits_{-\infty}^{\infty}\!\!\cdots\!\!\int f(p_1)f(p_2-p_1)\cdots f(p_k-p_{k-1})\times$$
$$\times X_k(H,p_k,\dots,p_1,s)dp_{k-1}\cdots dp_1\Big]+$$
$$+\hat{R}\Big[\frac{E}{(2\pi)^{2(k-1)}}\int\limits_{-\infty}^{\infty}\!\!\cdots\!\!\int f(p_1)f(p_2-p_1)\cdots f(p_k-p_{k-1})\times$$
$$\times X_{k-1}(H,p_{k-1},\dots,p_1,s)dp_{k-1}\cdots dp_1\Big]\Big)\vec{t}_H.$$

By comparing (71 - 73) and (68 - 70) we establish that assumption (66) is true, and occurs for all $k\geq 2$, while for the vector functions $\vec{\mathcal{P}}_k$, presentation (66) occurs through the corresponding vector function $\vec{\Psi}_k$. Thus, if the vector function $\vec{\Psi}_k$ satisfies the boundary value tasks 10, then the corresponding vector function $\vec{\mathcal{P}}_k$ is determined from formulas (62).

We now assume that for all k' = 2,3,...k, ratio (62) occurs. Then, by substituting in (63 - 65) instead of $\vec{\mathcal{P}}_k$, expression (62), and also /13 (67), we arrive at an equivalent boundary value task (71 - 73). With fixed values of parameters $P_1,\dots,P_k$, from this task we obtain the $\vec{\Psi}_k$ parametric set of boundary value tasks 10 for the vector function k, i.e., if presentation (62) occurs, then the vector functions $\vec{\Psi}_k$ satisfy boundary value tasks 10.

Solution to boundary value tasks 10 is determined only by regular parameters of the layer and backing model: $\varepsilon_{sc}(z)$, $\varepsilon_{ex}(z)$, γ .
Consequently, the vector functions $\vec{\Psi}_k$ are universal characteristics of the nonlinear transfer system of radiation with regular parameters, do not depend on the conditions of illumination and are invariant in relation to the horizontal heterogeneities of the albedo $q_r(\omega)$. The vector functions $\vec{\Psi}_k(z,p_k,\dots,p_1,s)$ are called vector SFC of the k-th order of approximation for parameter ε, corresponding to the term $\vec{\Phi}_k$ in the expansion (23). In summarizing what has been said, we obtain a calculated formula

$$\vec{\Phi}_k(r_\perp,z,s)=\frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\vec{\mathcal{P}}_k(z,p_k,s)e^{-i(p_k r_\perp)}dp_k\,\frac{E}{(2\pi)^{2k}}\int\limits_{-\infty}^{\infty}\!\!\cdots\!\!\int f(p_1)\times$$
$$\times f(p_2-p_1)\cdots f(p_k-p_{k-1})\vec{\Psi}_k(z,p_k,\dots,p_1,s)e^{-i(p_k r_\perp)}dp_k\cdots dp_1= \quad (74)$$

$$= \frac{E}{(2\pi)^{2k}} \int_{-\infty}^{\infty} \cdots \int \vec{\Psi}_k(z, p_k, \ldots, p_1, \theta)\, dp_k \ldots dp_1\, e^{-i(p_k, r_1)} \cdot$$
$$\cdot \int_{-\infty}^{\infty} q_p(r_{11})\, e^{i(p_1, r_{11})}\, dr_{11} \int_{-\infty}^{\infty} q_p(r_{12})\, e^{i(p_2 - p_1, r_{12})}\, dr_{12} \ldots \int_{-\infty}^{\infty} q_p(r_{1k}) \cdot$$
$$\cdot \int_{-\infty}^{\infty} q_p(r_{1k})\, e^{i(p_k - p_{k-1}, r_{1k})}\, dr_{1k} = \frac{E}{(2\pi)^{2k}} \int_{-\infty}^{\infty} \cdots \int \vec{\Psi}_k(z, p_k, \ldots, p_1, \theta)\, dp_k \ldots dp_1 \cdot$$
$$\cdot \int_{-\infty}^{\infty} \cdots \int q_p(r_{11}) \ldots q_p(r_{1k})\, e^{i\left[(p_k, r_{1k} - r_1) + \sum_{l=1}^{k-1} (p_l, r_{1l} - r_{1,l+1})\right]}\, dr_{11} \ldots dr_{1k}$$

Thus, if there is a known solution to the one-dimensional boundary value task 10, then any term in the perturbation series (23) can be computed through the vector functions $\vec{\Psi}_k$ using formulas (74).

We will return to boundary value task 10. In this task, the dependence on parameters $P_1, \ldots, P_{k-1}$ is introduced through the term in the boundary condition $\hat{R}\chi_{k-1}(H, p_{k-1}, \ldots, p_1, \theta)$. Since all the operators $\hat{R}$, $\hat{S}$ and $\hat{R}$ in boundary value task 10 do not act on parameters $P_1, \ldots, P_{k-1}$, then because of the linearity of the transfer equations, factorization by parameters occurs:

$$\vec{\Psi}_k(z, p_k, \ldots, p_1, \theta) = \vec{W}_k(z, p_k, \theta)\, \hat{R}\chi_{k-1}(H, p_{k-1}, \ldots, p_1, \theta) \qquad (75)$$

and consequently, (see [58])

$$\chi_k(z, p_k, \ldots, p_1, \theta) = \mathfrak{z}_k(z, p_k, \theta)\, \hat{R}\chi_{k-1}(H, p_{k-1}, \ldots, p_1, \theta). \qquad (76)$$ /14

By substituting (75) and (76) in boundary value task 10, we arrive at boundary value task 12 with one parameter P_k:

$$\hat{K}(p_k)\vec{W}_k(z, p_k, \theta) = \hat{S}\vec{W}_k;$$
$$\vec{W}_k\big|_0 = 0; \quad \vec{W}_k\big|_H = (q\hat{R}\mathfrak{z}_k(H, p_k, \theta) + 1)\vec{E}_H. \qquad (77 - 79)$$

But this boundary value task is adequate to boundary value task 8: in both tasks with the same integral $\hat{S}$, $\hat{R}$ and differential $\hat{K}(p)$ operators, the sources, coefficients and boundary conditions coincide. It follows from here that for all $k \geq 2$:

$$\vec{W}_k(z,p_k,s) \equiv \vec{\Psi}_1(z,p_k,s), \quad \chi_k(z,p_k,s) \equiv X_1(z,p_k,s). \tag{80 - 81}$$

with k = 2

$$\vec{\Psi}_2(z,p_2,p_1,s) = \vec{W}_2(z,p_2,s)\hat{R}X_1(H,p_1,s) = \vec{\Psi}_1(z,p_2,s)\hat{R}X_1 \tag{82}$$

$$X_2(z,p_2,p_1,s) = \chi_2(z,p_2,s)\hat{R}X_1(H,p_1,s) = X_1(z,p_2,s)\hat{R}X_1(H,p_1,s). \tag{83}$$

Similarly, after further continuing the recurrent link with the help of ratios (75 - 76), and (80 - 81) and

$$\hat{R}X_{k-1}(H,p_{k-1},\dots,p_1,s) = \prod_{\ell=1}^{k-1}\hat{R}X_1(H,p_\ell,s), \tag{84}$$

we obtain a presentation of the vector SFC of any order of approximation $k>2$ to the vector SFC of the linear approximation:

$$\vec{\Psi}_k(z,p_k,\dots,p_1,s) = \vec{\Psi}_1(z,p_k,s)\prod_{\ell=1}^{k-1}\hat{R}X_1(H,p_\ell,s) \tag{85}$$

This is an extremely important ratio, since it reduces solution to the recurrently linked boundary value tasks 10 with increasing number of parameters $P_1,\dots,P_k$ to a solution of one boundary value task 8 with one real parameter P. By substituting (85) into (74), we find an expression for the expansion term (23) through the linear SFC:

$$\begin{aligned}\vec{\Phi}_k(r_1,t,s) &= \frac{E}{(2\pi)^{2k}}\int\!\!\cdots\!\!\int \vec{\Psi}_1(z,p_k,s)\prod_{\ell=1}^{k-1}\hat{R}X_1(H,p_\ell,s)\,dp_k\dots dp_1\,\cdot\\ &\cdot\int\!\!\cdots\!\!\int q_1(r_{1k})\dots q_1(r_{1k})\cdot\exp\Big[i(p_k,r_{1k}-r_1)+\sum_{n=1}^{k-1}(p_{k-n},r_{1k-n}-r_{1k-n+1})\Big]\cdot\\ &\cdot dr_{1k}\dots dr_{11} = \frac{E}{(2\pi)^{2k}}\int\!\!\cdots\!\!\int f(p_1)f(p_2-p_1)\dots f(p_k-p_{k-1})\vec{\Psi}_1(z,p_k,s)\cdot\\ &\cdot e^{-i(p_k,r_1)}\prod_{\ell=1}^{k-1}\hat{R}X_1(H,p_\ell,s)\,dp_k\dots dp_1.\end{aligned} \tag{86}$$

As a result, the Stokes vector can be computed by summing the series /15 (23), assuming parameter $\varepsilon = 1$, in the form of superposition of three terms:

$$\vec{\Phi}(z, S) = \vec{\Phi}_0(z,S) + \vec{\Phi}_1(z,S) + \vec{\Phi}_2(z,S), \tag{87}$$

where $\vec{\Phi}_1(\tau_\perp, z, s)$ --linear approximation of (54); the component of nonlinear approximations, starting from second and further:

$$\vec{\Phi}_2(\tau_\perp, z, s) = \sum_{k=2}^{\infty} \frac{E}{(2\pi)^{2k}} \int \cdots \int_{-\infty}^{\infty} f(p_1) f(p_2 - p_1) \ldots f(p_k - p_{k-1}) e^{-i(p_k, \tau_\perp)} \cdot \vec{\Psi}_1(z, p_k, s) \prod_{l=1}^{k-1} \hat{R} X_1(H, p_l, s)\, dp_k \ldots dp_1. \tag{87a}$$

$\vec{\Phi}_0(z,S)$ --solution to the one-dimensional vector boundary value task 5; the quantity E (30) is determined through the first component $I_0(z,S)$ of vector $\vec{\Phi}_0(z,s)$; f(P)--Fourier pattern (32) of the variable component of the bottom albedo $q_p(\tau_\perp)$; the vector $\vec{\Psi}_1(z, p_k, s)$--solution to the one-dimensional vector boundary value task 8 with parameter P_k; the quantity $c_1(p_l) = \hat{R} X_1(H, p_l, s)$ is calculated by integration for the angular variable S (2) of the first component X_1 of vector $\vec{\Psi}_1$ only at the bottom level (z = H) on the hemisphere of directions $s \in \Omega^+$.

3. Vector Functions of Scattering, or Vector Functions of Influence

We will examine the series of perturbations (23). We will introduce the vector function of scattering [3, 4, 6] of the first order $\vec{\Theta}_1(z, \tau_\perp, s)$ so that

$$\vec{\Psi}_1(z, p, s) = \int_{-\infty}^{\infty} \vec{\Theta}_1(z, \tau_\perp, s) e^{i(p, \tau_\perp)} d\tau_\perp, \tag{88}$$

i.e., the vector scattering function $\vec{\Theta}_1$ is the inverse Fourier pattern of the vector linear SFC $\vec{\Psi}_1$

$$\vec{\Theta}_1(z, \tau_\perp, s) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \vec{\Psi}_1(z, p, s) e^{-i(p, \tau_\perp)} dp. \tag{89}$$

With the help of ratios [2]

$$(s_\perp, \frac{\partial \vec{\Phi}}{\partial \tau_\perp}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} [-i(p, s_\perp) \vec{\mathcal{P}}(z, p, s) e^{-i(p, \tau_\perp)}] dp, \tag{90}$$

$$\int_{-\infty}^{\infty} (s_\perp, \frac{\partial \vec{\Phi}}{\partial \tau_\perp}) e^{i(p, \tau_\perp)} d\tau_\perp = -i(p, s_\perp) \vec{\mathcal{P}}(z, p, s) \tag{91}$$

By inverse Fourier transform in boundary value task 8, we obtain boundary value task 13 for the vector scattering function of the linear approximation

$$\vec{\Theta}_1(z, \tau_\perp, s) = \{\Theta_{I1}, \Theta_{Q1}, \Theta_{U1}, \Theta_{V1}\}; \tag{92}$$

$$\hat{\mathcal{D}} \vec{\Theta}_1 = \hat{S} \vec{\Theta}_1; \quad \vec{\Theta}_1|_0 = 0; \quad \vec{\Theta}_1|_H = (q, \hat{R}\vec{\Theta}_{H1} + \delta(\tau_\perp)) \vec{T}_H . \tag{93 - 95}$$

By comparing this task with boundary value task 6, we are convinced that $\vec{\Theta}_1(z, \tau_\perp, s)$ is the Green vector function of boundary value task 6 for linear approximation of the Stokes vector $\vec{\Phi}(z, \tau_\perp, s)$. Or in terms of the monograph of V. S. Vladimirov [2], $\vec{\Theta}_1(z, \tau_\perp, s)$ is a fundamental solution, or a vector function of influence of the vector boundary value task 6 with perturbed boundary condition, more accurately, with isotropic δ-source on the boundary Z = H at the point with coordinate $\tau_\perp = 0$. /16

In order to conduct the numerical calculations, we give preference to the following algorithm: initially boundary value task 6 which is one-dimensional for space is numerically solved by the vector function $\vec{\Psi}_1(z, p, s)$ with a set of parameter values P; through it using formula (89) by the method of rapid Fourier transform, the vector function of scattering $\vec{\Theta}_1(z, \tau_\perp, s)$ is calculated. Numerical solution to boundary value task 13 directly for vector function $\vec{\Theta}_1(z, \tau_\perp, s)$ is difficult to realize on a computer, since this task contains five variables $(x, y, z, \vartheta, \varphi)$

However, in order to solve task of the boundary value type 13 containing local spot or spatially limited sources, method of SFC is also applicable which still has an advantage: in these tasks, the series of the perturbation theory of type (23) only contains linear terms. An individual publication will cover this problem.

The Stokes vector $\vec{\Phi}_\perp$ (54) with the help of (88) can be presented in the form of an integral packet:

$$\begin{aligned}\vec{\Phi}_\perp(r_\perp,z,s) &= \frac{E}{(2\pi)^2}\iint_{-\infty}^{\infty}\vec{\Theta}_\perp(r_{\perp 2},z,s)\cdot e^{i(p_\perp,r_\perp)}dr_{\perp 2}\, f(p_\perp)e^{-(p_\perp,r_\perp)}dp_\perp = \frac{E}{(2\pi)^2}\iiint_{-\infty}^{\infty}\vec{\Theta}_\perp(r_{\perp 2},z,s)e^{i(p_\perp,r_{\perp 2})}dr_{\perp 2}\cdot\\ &\cdot q_r(r_{\perp 1})e^{i(p_\perp,r_{\perp 1})}dr_{\perp 1}\,e^{-i(p_\perp,r_\perp)}dp_\perp = E\iint_{-\infty}^{\infty} q_r(r_{\perp 1})\,\vec{\Theta}_\perp(r_{\perp 2},z,s)\cdot\\ &\cdot\left[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}e^{i(p_\perp,r_{\perp 1}+r_{\perp 2}-r_\perp)}dp_\perp\right]dr_{\perp 1}\,dr_{\perp 2} = E\int_{-\infty}^{\infty}q_r(r_{\perp 1})dr_{\perp 1}\cdot\\ &\cdot\int_{-\infty}^{\infty}\vec{\Theta}_\perp(r_{\perp 2},z,s)\,\delta(r_{\perp 1}+r_{\perp 2}-r_\perp)dr_{\perp 2} = E\int_{-\infty}^{\infty}q_r(r_{\perp 1})\,\vec{\Theta}_\perp(r_\perp-r_{\perp 1},z,s)\cdot\\ &\cdot dr_{\perp 1} = E\int_{-\infty}^{\infty}q_r(r_\perp-r_{\perp 1})\,\vec{\Theta}_\perp(r_{\perp 1},z,s)\,dr_{\perp 1}.\end{aligned} \tag{96}$$

We will use replacement of variables $r'_{\perp 1}=r_\perp-r_{\perp 1}$, and also /17
formula [2]

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}e^{i(p,r)}dp = \delta(r). \tag{97}$$

We note in particular, that the following ratio follows from (88):

$$\vec{\Psi}_\perp^0(z,s) \equiv \vec{\Psi}_\perp(z,p=0,s) = \int_{-\infty}^{\infty}\vec{\Theta}_\perp(z,r_\perp,s)\,dr_\perp;\quad \vec{\Psi}_\perp^0 = \{X_\perp^0, Y_\perp^0, Z_\perp^0, T_\perp^0\}, \tag{98 - 99}$$

from which follows the existence and the limited nature of the integral in the right side of (98) because of the limited nature of the vector function $\vec{\Psi}_\perp^0(z,s)$. The vector function $\vec{\Psi}_\perp^0$ is real and is solution to the one-dimensional vector boundary value task 14:

$$\hat{D}_z\vec{\Psi}_\perp^0 = \hat{S}\vec{\Psi}_\perp^0;\quad \vec{\Psi}_\perp^0|_0 = 0;\quad \vec{\Psi}_\perp^0|_H = (q\hat{R}X_\perp^0+1)\vec{E}_H. \tag{100 - 102}$$

Evidently

$$\hat{R}X_{\perp}(H,p,s) = \int_{-\infty}^{\infty} \hat{R}\,\Theta_{X\perp}(H,\tau_{\perp},s)\, e^{i(p,\tau_{\perp})}\, d\tau_{\perp}, \tag{103}$$

$$\hat{R}\Theta_{X\perp}(\tau_{\perp},H,s) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{R}X_{\perp}(H,p,s)\, e^{-i(p,\tau_{\perp})}\, dp. \tag{104}$$

We introduce the vector functions of the k-th order of approximation of series (23):

$$\vec{\Theta}_k(z,\tau_{\perp k},\dots,\tau_{\perp 1},s) = \frac{1}{(2\pi)^{2k}} \int \cdots \int_{-\infty}^{\infty} \vec{\Psi}_k(z,p_k,\dots,p_1,s)\, e^{-i\sum_{n=1}^{k}(p_n,\tau_{\perp n})}\, dp_k \dots dp_1 \tag{105}$$

or

$$\vec{\Psi}_k(z,p_k,\dots,p_1,s) = \int \cdots \int_{-\infty}^{\infty} \vec{\Theta}_k(z,\tau_{\perp k},\dots,\tau_{\perp 1},s)\, e^{i\sum_{n=1}^{k}(p_n,\tau_{\perp n})}\, d\tau_{\perp k} \dots d\tau_{\perp 1}. \tag{106}$$

and we substitute (105) into (74):

$$\begin{aligned}
\vec{\Phi}_k(\tau_{\perp},z,s) &= \frac{E}{(2\pi)^{2k}} \int \cdots \int_{-\infty}^{\infty} f(p_1) f(p_2-p_1) \cdots f(p_k-p_{k-1})\, e^{-i(p_k,\tau_{\perp})} \cdot \\
&\quad \vec{\Theta}_k(z,\tau_{\perp 2k},\dots,\tau_{\perp k+1},s) \exp\Big[i\sum_{n=1}^{k}(p_n,\tau_{\perp k+n})\Big]\, dp_k \dots dp_1\, d\tau_{\perp 2k} \dots d\tau_{\perp k+1} = \\
&= E \underset{(3k)}{\int \cdots \int_{-\infty}^{\infty}} q_r(\tau_{\perp 1})\, e^{i(p_1,\tau_{\perp 1})}\, d\tau_{\perp 1}\, q_r(\tau_{\perp 2})\, e^{i(p_2-p_1,\tau_{\perp 2})}\, d\tau_{\perp 2} \dots q_r(\tau_{\perp k}) \cdot \\
&\quad \cdot e^{i(p_k-p_{k-1},\tau_{\perp k})}\, d\tau_{\perp k}\, \vec{\Theta}_k(z,\tau_{\perp 2k},\dots,\tau_{\perp k+1},s) \exp\Big[i\sum_{n=1}^{k}(p_n,\tau_{\perp k+n}) - \\
&\quad - i(p_k,\tau_{\perp})\Big]\, dp_k \dots dp_1\, d\tau_{\perp 2k} \dots d\tau_{\perp 1} = E \underset{(2k)}{\int \cdots \int_{-\infty}^{\infty}} q_r(\tau_{\perp 1}) \dots q_r(\tau_{\perp k}) \cdot \\
&\quad \cdot \vec{\Theta}_k(z,\tau_{\perp 2k},\dots,\tau_{\perp k+1},s) \cdot \Big[\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \exp[i(p_1,\tau_{\perp 1}-\tau_{\perp 2}+\tau_{\perp k+1})]\, dp_1\Big] \cdot \\
&\quad \cdot \Big[\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \exp[i(p_2,\tau_{\perp 2}-\tau_{\perp 3}+\tau_{\perp k+2})]\, dp_2\Big] \cdot \dots \cdot \Big[\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \exp[i(p_{k-1},\tau_{\perp k-1} - \\
&\quad - \tau_{\perp k} + \tau_{\perp 2k-1})]\, dp_{k-1}\Big] \cdot \Big[\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \exp[i(p_k,\tau_{\perp k}-\tau_{\perp}+\tau_{\perp 2k})]\, dp_k \cdot \\
&\quad \cdot d\tau_{\perp 1} \dots d\tau_{\perp 2k} =
\end{aligned} \tag{107}$$

$$= E\underbrace{\int\!\!\cdots\!\!\int_{-\infty}^{\infty}}_{(K)} q_r(\tau_1)\dots q_r(\tau_K)\,d\tau_K\dots d\tau_1 \underbrace{\int\!\!\cdots\!\!\int_{-\infty}^{\infty}}_{(K)} \vec{\Theta}_K(z, \tau_{12K},\dots,\tau_{1K+1}, s)\cdot$$
$$\cdot\,\delta(\tau_{11}-\tau_{12}+\tau_{1K+1})\,\delta(\tau_{12}-\tau_{13}+\tau_{1K+2})\dots\delta(\tau_{1K-1}-\tau_{1K}+\tau_{12K-1})\cdot$$
$$\cdot\,\delta(\tau_{1K}-\tau_1+\tau_{12K})\,d\tau_{12K}\dots d\tau_{1K+1} = E\int\!\!\cdots\!\!\int_{-\infty}^{\infty} q_r(\tau_1)\dots q_r(\tau_K)\cdot$$
$$\cdot\,\vec{\Theta}_K(z, \tau_1-\tau_K, \tau_K-\tau_{K-1},\dots,\tau_2-\tau_1, s)\,d\tau_K\dots d\tau_1 =$$
$$= E\int\!\!\cdots\!\!\int_{-\infty}^{\infty} q_r(\tau_2-\tau_1)\,q_r(\tau_3-\tau_2)\dots q_r(\tau_K-\tau_{K-1})\,q_r(\tau_1-\tau_K)\cdot$$
$$\cdot\,\vec{\Theta}_K(z, \tau_K,\dots,\tau_1, s)\,d\tau_K\dots d\tau_1.$$

It follows from (106) /18

$$\vec{\Psi}_K(z, \underbrace{0,\dots,0}_{K}, s) = \int\!\!\cdots\!\!\int_{-\infty}^{\infty} \vec{\Theta}_K(z, \tau_K,\dots,\tau_1, s)\,d\tau_K\dots d\tau_1. \qquad (108)$$

Using presentation (85) for the vector SFC $\vec{\Psi}_K$ and formulas (89), (103), and (104) for expression (105) we find the most convenient form, factorized, split for parameters P_K:

$$\vec{\Theta}_K(z, \tau_{1K},\dots,\tau_{11}, s) = \frac{1}{(2\pi)^{2K}}\int\!\!\cdots\!\!\int_{-\infty}^{\infty} \vec{\Psi}_1(z, p_K, s)\prod_{\ell=1}^{K-1}\hat{R}X_1(H, p_\ell, s)\cdot$$
$$\cdot\exp\left[-i\sum_{n=1}^{K}(p_n, \tau_n)\right]dp_K\dots dp_1 = \left[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\vec{\Psi}_1(z, p_K, s)\,e^{-i(p_K,\tau_K)}\,dp_K\right]\cdot \qquad (109)$$
$$\cdot\prod_{\ell=1}^{K-1}\left[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\hat{R}X_1(H, p_\ell, s)\,e^{-i(p_\ell,\tau_\ell)}\,dp_\ell\right] =$$
$$= \vec{\Theta}_1(\tau_K, z, s)\prod_{\ell=1}^{K-1}\hat{R}\Theta_{X1}(\tau_\ell, H, s).$$

As a result, the vector scattering function $\vec{\Theta}_K$ of any order of approximation $K \geq 2$ of the perturbation series (23) can be defined through the vector scattering function of linear approximation $\vec{\Theta}_1$ It is further easy to show that any term in the series (23) can be found using the integral packet:

$$\vec{\Phi}_K(\tau_1, z, s) = E\int\!\!\cdots\!\!\int_{-\infty}^{\infty} q_r(\tau_2-\tau_1)\dots q_r(\tau_K-\tau_{K-1})\cdot$$
$$\cdot\,q_r(\tau_1-\tau_K)\,\vec{\Theta}_1(\tau_K, z, s)\prod_{\ell=1}^{K-1}\hat{R}\Theta_{X1}(\tau_\ell, H, s)\,d\tau_K\dots d\tau_1 = \qquad (110)$$
$$= E\int_{-\infty}^{\infty} q_r(\tau_1)\dots q_r(\tau_K)\,\vec{\Theta}_1(\tau_1-\tau_K, z, s)\cdot$$
$$\cdot\prod_{\ell=1}^{K-1}\hat{R}\Theta_{X1}(\tau_{\ell+1}-\tau_\ell, H, s)\,d\tau_K\dots d\tau_1.$$

The Stokes vector $\vec{\Phi}$ can be computed by summing the series (23) similarly to (87), in which the linear approximation $\vec{\Phi}_1$ is computed according to formula (96), while /19

$$\vec{\Phi}_\Sigma(\tau_1, z, s) = E\sum_{k=2}^{\infty} \int \cdots \int q_p(\tau_{12}-\tau_{11}) \cdots$$
$$\cdots q_p(\tau_{1k}-\tau_{1k-1}) q_p(\tau_1-\tau_{1k}) \vec{\Theta}_1(\tau_{1k}, z, s) \prod_{l=1}^{k-1} \hat{R}\Theta_{x1}(\tau_{1l}, H, s) d\tau_{1k} \ldots d\tau_{11}. \quad (111)$$

The vector functions $\vec{\Theta}_k$ satisfy boundary value tasks 15, which are obtained after an inverse Fourier transform in boundary value task 10 using the relationships (90 - 91):

$$\hat{\mathcal{D}}(\tau_{1k}) \vec{\Theta}_k(z, \tau_{1k}, \ldots, \tau_{11}, s) = \hat{S}\vec{\Theta}_k \,; \quad \vec{\Theta}_k^{\downarrow}|_0 = 0,$$
$$\vec{\Theta}_k^{\uparrow}|_H = (q\hat{R}\Theta_{xk}(H, \tau_{1k}, \ldots, \tau_{11}, s) + \delta(\tau_{1k})\hat{R}\Theta_{xk-1}(H, \tau_{1k-1}, \ldots, \tau_{11}, s))\vec{t}_n, \quad (111a - 111c)$$

where the differential operator

$$\hat{\mathcal{D}}(\tau_{1k}) = \mu\frac{\partial}{\partial z} + \left(s_\perp, \frac{\partial}{\partial \tau_{1k}}\right) + \sigma_{tot}(z) \quad (111d)$$

acts on the variable τ_{1k}, while the variables $\tau_{1k-1}, \ldots, \tau_{11}$ in this task are parameters. We establish from here that factorization occurs coinciding with (109):

$$\vec{\Theta}_k(z, \tau_{1k}, \ldots, \tau_{11}, s) = \vec{\Theta}_1(z, \tau_{1k}, s)$$
$$\cdot \hat{R}\Theta_{xk-1}(H, \tau_{1k-1}, \ldots, \tau_{11}, s) = \vec{\Theta}_1(z, \tau_{1k}, s) \prod_{l=1}^{k-1} \hat{R}\Theta_{x1}(\tau_{1l}, H, s).$$

4. Calculation of the Constant Component of the Albedo in Vector SFC and Scattering Functions

We return to boundary value tasks for the vector SFC $\vec{\Psi}_1$. The multiplier in the boundary condition (53) only depends on the parameter P. We will designate it

$$t(p) = q\hat{R}X_1(H, p, s) + 1. \quad (112)$$

Because of the linearity of boundary value task 8, the vector function

$$\vec{W}(z,p,s) = \{W_x, W_y, W_z, W_\varphi\}, \tag{113}$$

introduced with the help of the ratio

$$\vec{\Psi}_L(z,p,s) = t(p)\,\vec{W}(z,p,s), \tag{114}$$

will satisfy boundary value task 16:

$$\hat{K}(p)\vec{W} = \hat{S}\vec{W};\quad \vec{W}|_0 = 0;\quad \vec{W}|_H = \vec{t}_H. \tag{115 - 117}$$

After a chain of several transforms:

$$X_L(H,p,s) = t(p)W_x(H,p,s);\quad \hat{R}X_L(H,p,s) = t(p)\hat{R}W_x(H,p,s),$$
$$t(p) = q\,t(p)\,\hat{R}W_x(H,p,s) + 1,\quad t(p) = 1/(1 - q\hat{R}W_x(H,p,s)) \tag{118 - 121}$$

we arrive at expression

$$\vec{\Psi}_L(z,p,s) = \frac{\vec{W}(z,p,s)}{1 - q\hat{R}W_x(H,p,s)} = \frac{\vec{W}(z,p,s)}{1 - q\,c(p)}. \tag{122}$$

as a result, the vector SFC $\vec{\Psi}_L$ of the linear system of transfer of radiation "atmosphere-uniform Lambert underlying surface with albedo q" is /20
analytically expressed through the linear SFC $\vec{W}$ of the isolated atmosphere. We note in this case that

$$c(p) = \hat{R}W_x(H,p,s) \tag{123}$$

is determined through the first vector component $\vec{W}$, and only depends on the spatial frequency P. Now the vector SFC (85) of any order $n \geq 2$ can be computed according to the formula

$$\vec{\Psi}_n(z,p_1,\dots,p_n,s) = \frac{\vec{W}(z,p_n,s)}{1 - q\,c(p_n)}\prod_{l=1}^{n-1}\frac{c(p_l)}{1 - q\,c(p_l)}. \tag{124}$$

Factorization for parameters is a convenient circumstance for realizing numerical calculations reducing the volume of computations. The components of the Stokes vector (87) can be computed in this case by merely knowing the vector SFC of the linear approximation W for the isolated atmosphere:

$$\vec{\Phi}_1(\tau,z,s)=\frac{E}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{f(p)\vec{W}(z,p,s)e^{-i(p,\tau)}}{1-qc(p)}dp,$$

$$\vec{\Phi}_\Sigma(\tau,z,s)=\sum_{k=2}^{\infty}\frac{E}{(2\pi)^{2k}}\int\!\!\cdots\!\!\int f(p_1)f(p_2-p_1)\cdots f(p_k-p_{k-1})\exp[-i(p_k,\tau)]\cdot$$
$$\cdot\frac{\vec{W}(z,p_k,s)}{1-qc(p_k)}\prod_{l=1}^{k-1}\frac{c(p_l)}{1-qc(p_l)}. \qquad (125-126)$$

It is apparent from these formulas that dependence of a Stokes vector on the constant component of albedo is not linear. However, it is easy to obtain from these same formulas an evaluation of the deviation from the linear relationship, by distributing the fractions 1/1-qc(P) into a Taylor series according to the degrees of argument qc(P). Using (122) for the vector scattering function $\vec{\Theta}_1$, one can also obtain a formula of linkage to the constant albedo component:

$$\vec{\Theta}_1(z,\tau,s)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\vec{W}(z,p,s)}{1-qc(p)}e^{-i(p,\tau)}dp. \qquad (127)$$

After this, the expression for $\vec{\Theta}_k$ can be written as

$$\vec{\Theta}_k=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\vec{W}(z,p_k,s)}{1-qc(p_k)}e^{-i(p_k,\tau_{1k})}dp_k\cdot\prod_{n=1}^{k-1}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{c(p_n)}{1-qc(p_n)}e^{-i(p_n,\tau_{1n})}dp_n. \qquad (128)$$

However, in this case, we do not succeed in presenting the Stokes vector in the form of an integral packet similar to (107) which would include only the scattering function determined for the isolated layer.

We introduce the vector scattering function $\vec{\Theta}'_k=\{\Theta'_{Ik},\Theta'_{Qk},\Theta'_{Uk},\Theta'_{Vk}\}$ through the SFC $\vec{W}_k=\{W_{Ik},W_{Qk},W_{Uk},W_{Vk}\}$, taken with zero value of q:

$$\vec{W}_k(z,p_k,\ldots,p_1,s)=\int\!\!\cdots\!\!\int_{-\infty}^{\infty}\vec{\Theta}'_k\exp\Big[i\sum_{n=1}^{k}(p_n,\tau_n)\Big]d\tau_k\ldots d\tau_1.$$

Starting from (122) and (124), one can write

$$\vec\Psi_K(z,p_K,\dots,p_1,s) = \frac{\vec W_K(z,p_K,\dots,p_1,s)}{\prod_{l=1}^{K-1}(1-qc(p_l))}, \quad \vec W_K(z,p_K,\dots,p_1,s)=\vec W(z,p_K,s)\prod_{l=1}^{K-1}c(p_l).$$

For the vector scattering function $\vec\Theta'_K = \{\theta'_{1K},\theta'_{2K},\theta'_{3K},\theta'_{4K}\}$ we have:

$$\vec\Theta'_1(z,\tau_{11},s) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\vec W(z,p_1,s)\exp[-i(p_1,\tau_1)]\,dp_1,$$

$$\vec\Theta'_K(z,\tau_K,\dots,\tau_1,s) = \vec\Theta'_1(z,\tau_K,s)\prod_{l=1}^{K-1}\hat R\theta'_{xl}(H,\tau_l,s).$$

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We use expression [74]

$$\begin{aligned}
\vec\Phi_K(\tau_1,z,s) &= \frac{E}{(2\pi)^{2K}}\int\cdots\int_{-\infty}^{\infty} f(p_1)f(p_2-p_1)\cdots f(p_K-p_{K-1})e^{-i(p_K,\tau_1)}\cdot\\
&\cdot\frac{\vec W_K(z,p_K,\dots,p_1,s)}{\prod_{l=1}^{K-1}(1-qc(p_l))}\,dp_K\cdots dp_1 = \frac{E}{(2\pi)^{2K}}\int\cdots\int_{-\infty}^{\infty}\frac{f(p_1)f(p_2-p_1)\cdots f(p_K-p_{K-1})}{\prod_{l=1}^{K-1}(1-qc(p_l))}\cdot\\
&\cdot\exp(-i(p_K,\tau_1))\,dp_K\cdots dp_1\int\cdots\int_{-\infty}^{\infty}\vec\Theta'_K(z,\tau_K,\dots,\tau_1,s)\exp\Big[i\sum_{l=1}^{K}(p_l,\tau_l)\Big]\cdot\\
&\cdot d\tau_{1K}\cdots d\tau_{11} = \frac{E}{(2\pi)^{2K}}\int\cdots\int_{-\infty}^{\infty}\frac{f(p_1)f(p_2-p_1)\cdots f(p_K-p_{K-1})}{\prod_{l=1}^{K-1}(1-qc(p_l))}e^{-i(p_K,\tau_1)}dp_K\cdots dp_1\cdot\\
&\cdot\int\cdots\int_{-\infty}^{\infty}\vec\Theta'_1(\tau_K)\prod_{l=1}^{K-1}\hat R\theta'_{xl}(\tau_l)\exp\Big[i\sum_{l=1}^{K}(p_l,\tau_l)\Big]d\tau_K\cdots d\tau_1 =\\
&= E\int\cdots\int_{-\infty}^{\infty} q_r(\tau_{1K+1})\cdots q_r(\tau_{2K})\,\vec\Theta'_1(\tau_K)\prod_{l=1}^{K-1}\hat R\theta'_{xl}(\tau_l)\,d\tau_{1K}\cdots d\tau_{11}\cdot\\
&\cdot\Big[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{1-qc(p_1)}e^{i(p_1,\tau_{1K+1}-\tau_{K+2}+\tau_1)}dp_1\Big]\cdot\ldots\cdot\Big[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{1-qc(p_{K-1})}\cdot\\
&\cdot e^{i(p_{K-1},\tau_{2K-1}-\tau_{2K}+\tau_{K-1})}dp_{K-1}\Big]\cdot\Big[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{1-qc(p_K)}e^{i(p_K,\tau_{2K}+\tau_K-\tau_1)}dp_K\Big].
\end{aligned}$$

This is not reduced to an expression of the packet, since the integrals for the parameters contain functions which differ from the constant, and therefore they are not taken analytically and do not result in δ-functions, as occurred, for example, in the case of derivation of formula (107).

Here the conclusion is suggested that a sequence of calculations be organized in the case of nonzero component of albedo q (if q = 0, then the just derived formula is reduced to formula (107)). Initially we calculate the vector linear SFC $\vec{W}$ as a solution to the boundary value task 16. Through it we compute the vector SFC $\vec{\Psi}_1$ according to formula (122) and the linear vector scattering function $\vec{\Theta}_1$ according to formula (127), while the corresponding nonlinear approximations are defined from formulas (124) or (85) for $\vec{\Psi}_K$, and according to formulas (128) or (109) for $\vec{\Theta}_K$.

5. Calculation of Diffuse Reflection from a Uniform Lambert Surface in Solving the Vector Transfer Equation

From the concepts obtained above for solving the vector boundary value task 1 with horizontally nonuniform boundary condition, described /22 by the Lambert reflection law (1), one particular, but extremely important result follows. We will examine boundary value task 5. It differs from boundary value task 3 because in the latter, the albedo is horizontally nonuniform. We did not place any restrictions in presenting albedo $q_N(r_\perp)$ in the form of (9), except one: there should be an integral Fourier transform (31) for the component $q_r(r_\perp)$. We place in (9) q = 0, while $q_r(r_\perp) = q_r = const$. In this case the Fourier pattern

$$f(p) = \mathcal{F}[q_r](p) = \int_{-\infty}^{\infty} q_r \exp[i(p, r_\perp)]\, dr_\perp = (2\pi)^2 q_r \delta(p), \tag{129}$$

The linear vector SFC

$$\vec{\Psi}_1(z, p, s) \equiv \vec{W}(z, p, s). \tag{130}$$

The linear approximation of the Stokes vector (125) looks like

$$\vec{\Phi}_1(z, s) = E q_r \int_{-\infty}^{\infty} \delta(p_\perp) \vec{W}(z, p_\perp, s) e^{-i(p_\perp, r_\perp)} dp_\perp = E q_r \vec{W}(z, p=0, s) = E q_r \vec{W}(z, s) \tag{131}$$

The formula of nonlinear approximations of Stokes vector (86) is simplified:

$$\vec{\Phi}_K(z,s) = E q_r^K \int\!\cdots\!\int \delta(p_1)\delta(p_2-p_1)\cdots\delta(p_K-p_{K-1})\vec{W}(z,p_K,s)\prod_{\ell=1}^{K-1} c(p_\ell)\cdot \exp[-i(p_K,\tau)]\,dp_1\ldots dp_K = E q_r^K c^{K-1}(p=0)\vec{W}(z,p=0,s) = E q_r^K c_0^{K-1}\vec{W}_0(z,s) \qquad (132)$$

The vector function

$$\vec{W}_0(z,s) = \vec{W}(z,p=0,s) = \{I_W, Q_W, U_W, V_W\} \qquad (133)$$

satisfies the standard one-dimensional vector boundary value task 17 with isotropic source of nonpolarized radiation on the lower boundary:

$$\hat{D}_z\vec{W}_0 = \hat{S}\vec{W}_0;\ \vec{W}_0|_0 = 0;\ \vec{W}_0|_H = \vec{E}_H. \qquad (134 - 136)$$

According to the principle of the maximum (35) and the properties of the operator $\hat{R}$, the value

$$c_0 = c(p=0) = \hat{R} I_W(H,s) \le 1. \qquad (137)$$

We will sum series (23) in the form:

$$\vec{\Phi} = \vec{\Phi}_0^0(z,s) + \vec{\Phi}_q(z,s), \qquad (138)$$

$\vec{\Phi}_0^0 = \{I_0^0, Q_0^0, U_0^0, V_0^0\}$ --solution to the boundary value task 5 with albedo q = 0,

$$\vec{\Phi}_q = \sum_{K=1}^{\infty}\vec{\Phi}_K(z,s) = E q_r \vec{W}_0(z,s)\sum_{K=0}^{\infty} q_r^K c_0^K = \frac{E q_r \vec{W}_0(z,s)}{1 - q_r c_0}. \qquad (139)$$

The series for $\vec{\Phi}_q$ was summed as a geometric progression with denominator $q_r c_0 < 1$, since usually $q_r < 1$. The value of the magnitude of illumination $\vec{\Phi}_q$ because of diffuse reflection from the uniform Lambert surface can be obtained by another method, based on presentation of terms in the series (23) through the vector scattering function. With $q_n(\tau)$ = q = const, we find from (96) with the involvement of (98):

$$\vec{\Phi}_1(z,s) = Eq_r \int_0^{\infty} \vec{\theta}_1(\tau_1, z, s)\, d\tau_1 = Eq_r\, \vec{\psi}_1(z, p=0, s) = Eq_r\, \vec{\psi}_1^{\,\circ}(z,s). \tag{140}$$

from (107) and ratio

$$\vec{\psi}_k(z, \underbrace{0, \ldots, 0}_{k}, s) = \int_0^{\infty} \!\!\cdots\! \int_0^{\infty} \vec{\theta}_k(z, \tau_k, \ldots, \tau_1, s)\, d\tau_k \ldots d\tau_1 \tag{141}$$

following from (106), we find that

$$\vec{\Phi}_k(z,s) = \vec{\Phi}_k(\tau_0, z, s) = \\ = Eq_r^{\,k} \int_0^{\infty}\!\!\cdots\!\int_0^{\infty} \vec{\theta}_k(z, \tau_k, \ldots, \tau_1, s)\, d\tau_k \ldots d\tau_1 = Eq_r^{\,k}\, \vec{\psi}_k(z, \underbrace{0, \ldots, 0}_{k}, s). \tag{142}$$

But according to (85): /23

$$\vec{\psi}_k(z, \underbrace{0, \ldots, 0}_{k}, s) = \vec{\psi}_1^{\,\circ}(z,s)\left(c_1^{\circ}\right)^{k-1}, \tag{143}$$

where the real vector function $\vec{\psi}_1^{\,\circ}$ satisfies the standard vector boundary value task 18 obtained from boundary value task 8 with value of the parameter P = 0:

$$\hat{D}_z \vec{\psi}_1^{\,\circ}(z,s) = \hat{S}\vec{\psi}_1^{\,\circ}, \\ \vec{\psi}_1^{\,\circ}|_0 = 0, \quad \vec{\psi}_1^{\,\circ}|_H = (q\, \hat{R}\chi_1^{\circ}(H, p=0, s) + 1)\vec{t}_H, \tag{144 - 146}$$

while the quantity

$$c_1^{\circ} = \hat{R}\chi_1^{\circ}(H,s) \tag{147}$$

We will sum the series (23):

$$\vec{\Phi}_{q_r} = \sum_{k=1}^{\infty} \vec{\Phi}_k(z,s) = Eq_r \vec{\psi}_1^{\,\circ}(z,s) \sum_{k=0}^{\infty} q_r^{\,k} \left(c_1^{\circ}\right)^k = \frac{Eq_r\, \vec{\psi}_1^{\,\circ}(z,s)}{1 - q_r c_1^{\circ}}. \tag{148}$$

The boundary value task 18 is a particular case of boundary value task 8 with P = 0. Consequently, for vector $\vec{\psi}_1^{\,\circ}$, the same properties are correct as for $\vec{\psi}_1$. In particular, similarly to (114) we have:

$$\vec{\psi}_1^{\circ}(z,s) = t_0 \vec{W}_0(z,s), \quad \vec{\psi}_1^{\circ}(z,s) = \frac{1}{1-qc_0} \vec{W}_0(z,s), \quad t_0 = \frac{1}{1-qc_0}. \tag{149 - 151}$$

Thus,

$$\vec{\Phi}_{q_r} = \frac{E q_r \vec{W}_0(z,s)}{1-(q+q_r)c_0}. \tag{152}$$

Assuming q = 0 (compare (152) and (139)), we obtain

$$\vec{\Phi}_{q_r} \equiv \vec{\Phi}_q. \tag{153}$$

It is appropriate to note here that the obtained formulas for calculating the horizontally uniform albedo component result in two important conclusions. First of all, for calculations of the Stokes vector $\vec{\Phi}$ of the atmosphere for the uniform Lambert surface, it is sufficient to solve the vector boundary value task 5 for the vector $\vec{\Phi}_0(z,s)$ with zero boundary conditions (in (29), we assume q = 0), i.e., for an isolated atmosphere, and then we take into consideration the component of the Stokes vector $\vec{\Phi}_q$ (139) governed by illumination because of diffuse reflection from the bottom. In this case, the quantity $\mathcal{J}E$ is computed through the first component of Stokes vector $\vec{\Phi}_0$ and parameters of external parallel stream on the upper boundary (30), and determine the illumination of the atmosphere bottom. The vector function $\vec{W}_0(z,s)$ does not depend either on the illumination conditions, or on the albedo of the bottom, as a universal characteristic of the atmosphere itself, essentially describing the propagation in the atmosphere with regard for polarization of diffuse natural nonpolarized light emitted by the uniform backing. Under conditions of observation from above, the function $\vec{W}_0$ practically coincides with the characteristics of atmospheric transmission with regard for the effects of polarization and multiple scattering. The coefficient C_0 is an analog of the spherical albedo of nonuniform atmosphere illuminated from below by parallel beams (9). It is obtained as a result of integration of the descending radiation of all directions of dissemination $s \in \Omega^+$, and in all directions of incidence of the external stream from below $s_0 \in \Omega^-$ on the boundary Z = H. Secondly, from the obtained results,

our hypothesis regarding the fairly random solution of the constant component q in presentation of the albedo in form (9) is confirmed: in addition to the quantity q, a certain constant component can also be contained in what we call the albedo variation, i.e., in the quantity $q_r(\tau_\perp)$. Its contribution is taken into consideration analytically using the component of Stokes vector $\vec{\Phi}_{q_r}$ according to formula (152). /24

6. Evaluations of Nonlinear Approximations

Using the presentations (96) and (107) of the terms in series (23) through integral packets of albedo variations with vector scattering functions, it is easy to obtain evaluations of the contribution of nonlinear terms with numbers $k \geq k_1$, $k_1 \geq 2$,--any assigned number. If we assume

$$0 \leq q_{sup} = \max|q_r(\tau_\perp)| < 1 \qquad (154)$$

then from (107) with regard for (105), (85) and (143) we have:

$$\begin{aligned} \vec{\Phi}_k(\tau_\perp, z, s) &\leq E q_{sup}^k \underbrace{\int \cdots \int}_{} \vec{\Theta}_k(z, \tau_\perp - \tau_{\perp k}, \tau_{\perp k} - \tau_{\perp k-1}, \ldots, \tau_{\perp 2} - \tau_{\perp 1}, s)\, d\tau_{\perp k} \ldots d\tau_{\perp 1} = \frac{E q_{sup}^k}{(2\pi)^{2k}} \underbrace{\int \cdots \int}_{} \vec{\Psi}_k(z, p_k, \ldots, p_1, s) \cdot \\ &\cdot \exp[-i[(p_k, \tau_\perp - \tau_{\perp k}) + (p_{k-1}, \tau_{\perp k} - \tau_{\perp k-1}) + \cdots + (p_1, \tau_{\perp 2} - \tau_{\perp 1})]]\, d\tau_{\perp k} \ldots \\ &\ldots d\tau_{\perp 1}\, dp_k \ldots dp_1 = E q_{sup}^k \vec{\Psi}_k(z, \underbrace{0, \ldots, 0}_{k}, s) = \\ &= E q_{sup}^k \vec{\Psi}_1^{\circ}(z, s) (\hat{R} X_1^{\circ}(H, s))^{k-1}. \end{aligned} \qquad (155)$$

Evaluation from above of the contribution of nonlinear approximation (23) $: \Delta\vec{\Phi}_{k_1} =$

$$= \sum_{k=k_1}^{\infty} \varepsilon^k \vec{\Phi}_k \leq E \vec{\Psi}_1^{\circ}(z, s) \sum_{k=k_1}^{\infty} q_{sup}^k (\hat{R} X_1^{\circ})^{k-1} = \frac{q_{sup}^{k_1} E \vec{W}_0(z, s) C_0^{k_1 - 1}}{(1 - qC_0)^{k_1 - 1}(1 + (q + q_{sup})C_0)}. \qquad (156)$$

With $q = 0$, the formula is simplified

$$\Delta\vec{\Phi}_{k_1} \leq q_{sup}^{k_1} E \vec{W}_0(z, s) C_0^{k_1 - 1} / (1 - q_{sup} C_0). \qquad (157)$$

It is important that evaluations of nonlinear approximations are expressed through the characteristics of an isolated atmosphere, which can

be calculated using the simplest boundary value tasks which have already been discussed above. With $k_1 = 1$, we obtain from (156) the upper evaluation for complete solution to the vector boundary value task 3:

$$\vec{\Phi}(z,s) \leq \vec{\Phi}_0(z,s)\Big|_{q=0} + \frac{Eq\,\vec{W}_0(z,s)}{1-qC_0} + \frac{q_{sup}\,E\,\vec{W}_0(z,s)}{1-(q+q_{sup})C_0}, \tag{158}$$

where $\vec{\Phi}_0(z,s)\big|_{q=0}$ --solution to boundary value task 5 with zero boundary conditions.

7. Fundamental Solution to the Vector of Transfer Equation with Local Perturbation in the Boundary Condition

If we take $q_r(r_\perp) = \delta(r_\perp)$, then f(P) = 1. We find the expression for the corresponding Stokes vector $\vec{\Phi}_k$, based on the presentation (74) and determination of (106): /25

$$\begin{aligned} \vec{\Phi}_k(r_\perp, z, s) &= \frac{E}{(2\pi)^{2k}} \int_{-\infty}^{\infty}\!\!\cdots\!\int e^{-i(p_k, r_\perp)} \cdot \\ &\cdot dp_k \ldots dp_1\, \vec{\Theta}_k(z, r_{\perp k}, \ldots, r_{\perp 1}, s) \exp\Big[i\sum_{l=1}^{k}(p_l, r_{\perp l})\Big] dr_{\perp k} \ldots dr_{\perp 1} = \\ &= E\int_{-\infty}^{\infty}\!\!\cdots\!\int \vec{\Theta}_k(z, r_{\perp k}, \ldots, r_{\perp 1}, s)\, \delta(r_{\perp 1}) \cdots \delta(r_{\perp k-1})\, \delta(r_{\perp k} - r_\perp)\, dr_{\perp k} \ldots dr_{\perp 1} = \\ &= E\,\vec{\Theta}_k(z, r_\perp, \underbrace{0, \ldots, 0}_{k-1}, s) = E\,\vec{\Theta}_1(r_\perp, z, s)(C_2^0)^{k-1}. \end{aligned} \tag{159}$$

Here the following designations have been introduced:

$$\begin{aligned} C_2(r_{\perp l}) &= \hat{R}\Theta_{x1}(r_{\perp l}, H, s) = \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} C_1(p_l) \exp[-i(p_l, r_{\perp l})]\, dp_l; \quad C_2^0 = C_2(r_{\perp l} = 0) = \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} [(\hat{R}W_x(H, p_l, s))/(1 - qC(p_l))]\, dp_l = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \frac{C(p_l)\, dp_l}{1 - qC(p_l)} \end{aligned} \tag{160 - 161}$$

Based on presentation of the vector $\vec{\Phi}_k$ in the form (110), after simple transforms, we arrive at the same result:

$$\begin{aligned} \vec{\Phi}_k(r_\perp, z, s) &= \\ &= E\int_{-\infty}^{\infty} \delta(r_{\perp k})\, \vec{\Theta}_1(r_\perp - r_{\perp k})\, dr_{\perp k} \int_{-\infty}^{\infty} \delta(r_{\perp k-1})\, \hat{R}\Theta_{x1}(H, r_{\perp k} - r_{\perp k-1}, s) \cdot \ldots \cdot \\ &\cdot \int_{-\infty}^{\infty} \delta(r_{\perp 1})\, \hat{R}\Theta_{x1}(H, r_{\perp 2} - r_{\perp 1}, s)\, dr_{\perp 1} = E\,\vec{\Theta}_1(r_\perp, z, s) \prod_{l=1}^{k-1} \hat{R}\Theta_{x1}(H, r_{\perp l} = 0, s) = \\ &= E\,\vec{\Theta}_k(z, r_\perp, 0, \ldots, 0, s). \end{aligned} \tag{162}$$

It follows from this that the vector scattering function $\vec{\Theta}_1(z,\tau_1,s)$ and $\vec{\Theta}_k(z,\tau_1,0,\ldots,0,s)$ are fundamental solutions to boundary value tasks 6 and 9 respectively with δ-source on the lower boundary. As a result of summing series (23), we obtain an expression for the fundamental solution to boundary value task 3:

$$\vec{\Phi}_\delta(\tau_1,z,s)=\vec{\Phi}_0(z,s)+E\sum_{k=1}\vec{\Theta}_k(z,\tau_1,0,\ldots,0,s)=\vec{\Phi}_0(z,s)+E\vec{\Theta}_1(z,\tau_1,s);\ c(p)=c(-p)\Rightarrow C_2^0=0. \tag{163}$$

8. Amplitude and Phase Characteristics of the Vector SFC

It follows from what has been said above that in order to solve the vector boundary task 3 with regard for any order of nonlinear approximations of the perturbation series (23), it is sufficient to solve the task for the vector SFC of linear approximation $\vec{W}(z,p,s)$ for an isolated layer, i.e., initially solve the parametric boundary value task 16, and then calculate $\vec{\Psi}_1$ according to formula (122). We introduce the amplitude and phase characteristics through the diffuse component $\vec{\mathcal{X}}$ of the vector SFC $\vec{\Psi}_1$:

$$\vec{\Psi}_1(z,p,s)=\begin{cases}\vec{\mathcal{X}},\ s\in\Omega^+;\\ e^{i(p,\vec{\tau})}(\Lambda^{-1}\vec{I}_H+\vec{\mathcal{X}}),\ s\in\Omega^-;\end{cases}\quad \vec{\mathcal{X}}=\{\mathcal{X}_m\};\ \mathcal{X}_m=A_m e^{i\varphi_m},\ m=1,2,3,4. \tag{164}$$

The vector AFC is $\vec{A}(z,p,s)=\{A_m\}$; $\vec{\Phi}(z,p,s)=\{\varphi_m\}$ --vector FFC:

$$A_m=\sqrt{(\mathrm{Re}\,\mathcal{X}_m)^2+(\mathrm{Im}\,\mathcal{X}_m)^2};\quad \varphi_m=\mathrm{arctg}[\mathrm{Im}\,\mathcal{X}_m/\mathrm{Re}\,\mathcal{X}_m],\ \mathcal{X}_m\neq 0. \tag{165}$$

It is easy to establish that

$$\vec{\Phi}(z,p,s)|_{p=0}=0;\quad \lim_{|p|\to\infty}A_m(z,p,s)=0.\quad \vec{\Phi}(z,p,\vec{s})|_{|\mu|=1}=0 \tag{166}$$

with observation in the nadir of no phase distortions. $\vec{\tau}_1=\frac{(H-z)s_1}{|\mu|}$ -- vector of shift in the horizontal plane with inclined sighting routes:

$\Lambda^{-1} = e^{-\frac{\tau_H - \tau(H)}{|\mu|}}$ --Bouguer attenuation. The vector $\vec{A}^{\circ} = \left\{ \frac{A_m(p)}{A_m(0)} \right\}$, which /26
is determined for those components with number m for which $A_m(0) \neq 0$ coincides with the frequency-contrast characteristics FCC, which only takes into consideration the amplitude distortions, $A^{\circ}_m(P = 0) = 1$. The introduction of nonlinear AFC and FFC is surplus, since all the nonlinear SFC $\vec{\psi}_k$ are expressed through linear SFC $\vec{\psi}_L$. The introduction of AFC and FFC for the vector $\vec{W}$ is less convenient, since the calculated formulas become less graphic, since c(p) is not always a real function.

Sometimes it is convenient to introduce the scattering function thrugh the diffuse part of the linear SFC:

$$\vec{\Theta}(\tau_\perp, z, s) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \vec{X}(z, p, s) e^{-i(p, \tau_\perp)} dp, \tag{167}$$

$$\vec{X}(z, p, s) = \int_{-\infty}^{\infty} \vec{\Theta}(\tau_\perp, z, s) e^{i(p, \tau_\perp)} d\tau_\perp ; \quad \vec{\Theta} = \{\Theta_x, \Theta_y, \Theta_z, \Theta_T\}. \tag{168}$$

The complete linear scattering function then is

$$\begin{aligned} \vec{\Theta}_L(z, \tau_\perp, s) &\equiv \vec{\Theta}_L(\tau_\perp - \tilde{\tau}_\perp, z, s) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} [\Lambda^{-1} \vec{t}_H + \vec{X}] e^{-i(p, \tau_\perp - \tilde{\tau}_\perp)} dp = \Lambda^{-1} \delta(\tau_\perp - \tilde{\tau}_\perp) \vec{t}_H + \\ &+ \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} X(p) e^{-i(p, \tau_\perp - \tilde{\tau}_\perp)} dp = \Lambda^{-1} \delta(\tau_\perp - \tilde{\tau}_\perp) \vec{t}_H + \vec{\Theta}(\tau_\perp - \tilde{\tau}_\perp, z, s). \end{aligned} \tag{169}$$

9. Direct Transfer Operator of the Atmosphere with Regard for Radiation Polarization

In summarizing, based on the results obtained above, one can write an expression in which the Stokes vector of radiation reflected by the atmosphere-underlying surface system will be analytically linked to the albedo

$$\vec{\Phi}(\tau_\perp, z, s) = \vec{\Pi}(q_H) = \vec{\Phi}_0^{\circ}(z, s) + \vec{\Phi}_q(z, s) + \vec{\Phi}_L(\tau_\perp, z, s) + \vec{\Phi}_\Sigma(\tau_\perp, z, s) \tag{170}$$

$\vec{\Phi}_0^{\circ}(z, s) \equiv \vec{\Phi}_0(z, s)|_{q=0}$ --solution to boundary value task 5 with zero conditions. Illumination because of constant albedo component q (139) is

computed through the quantity for isolated atmosphere:

$$\vec{\Phi}_q(z,s) = \frac{E_0 q \vec{W}_0(z,s)}{1-q c_0}; \quad E_0(\mu_0) = \hat{R} I_0^*(H,s) + S_\lambda \mu_0 e^{-\tau_0/\mu_0} \pi \quad (171 - 172)$$

--illumination of the bottom with incidence of parallel rays on the upper boundary in direction S_0; $c_0 = \hat{R} I_0^{(H,s)}$ --spherical albedo; $W_0(Z,S) = \Lambda^{-1}\vec{\xi}_n + \vec{\nabla}\vec{\Phi}(z,s)$ --vector transmission of isolated atmosphere illuminated by parallel rays from below. Illumination πE (30) with $q \neq 0$:

$$E = (\hat{R} I_0^* + f_n) + \frac{E_0 q \hat{R} I_w}{1-q c_0} = \frac{E_0}{1-q c_0}. \quad (173)$$

The linear approximation for calculating the horizontal heterogeneities of the albedo (54):

$$\vec{\Phi}_1(r_\perp - \xi, z, s) = \frac{E_0}{1-q c_0}\{\Lambda^{-1} q_r (r_\perp - \xi_\perp) f_0 + \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} f(p)\vec{I}(p) e^{i(p, \xi_\perp - r_\perp)} dp\} = \frac{E_0}{1-q c_0}\{\Lambda^{-1} q_r (r_\perp - \xi_\perp) + \int_{-\infty}^{\infty} q_r(r_{\perp 1}) \vec{\Theta}(r_\perp - \xi_\perp - r_{\perp 1}) dr_{\perp 1}\}. \quad (174)$$

The contribution of nonlinear approximation $\vec{\Phi}_\Sigma = \sum_{k=2} \vec{\Phi}_k$ is computed /27 according to formula (111) through the linear vector scattering function $\vec{\Theta}_1$, or according to formula (87a) through the vector linear SFC $\vec{\Psi}_1$ Upper estimates of the contribution of nonlinear approximations are obtained from formula (156). The direct transfer optical operator of the atmosphere (170) $\vec{\Pi}(q_n)$ is a generalization of the similar operator for the scalar task [13].

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